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COBOUNDARIES OF IRREDUCIBLE MARKOV OPERATORS ON $C(K)$

BY

ISAAC KORNFELD

Department of Mathematics, North Dakota State University
Fargo, ND 58105, USA
e-mail: kornfeld@plains.nodak.edu

AND

MICHAEL LIN

Department of Mathematics, Ben-Gurion University of the Negev
Beer Sheva 84105, Israel
e-mail: lin@black.bgu.ac.il

ABSTRACT

Let K be a compact Hausdorff space, and let T be an irreducible Markov operator on $C(K)$. We show that if $g \in C(K)$ satisfies $\sup_N \|\sum_{j=0}^N T^j g\| < \infty$, then (and only then) there exists $f \in C(K)$ with $(I-T)f = g$. Generalizing the result to irreducible Markov operator representations of certain semi-groups, we obtain that bounded cocycles are (continuous) coboundaries. For minimal semi-group actions in $C(K)$, no restriction on the semi-group is needed.

1. Coboundaries of a single irreducible Markov operator

In the classical Poisson equation, for a given function g we look for a function f such that $\Delta f = g$ (where Δ is the Laplacian). This equation can be given the abstract form of studying the range of the generator of a contraction semi-group in a Banach space (e.g., [KL]). The discrete abstract Poisson equation for a contraction T in a Banach space $\mathbb{B}$ is to solve $(I-T)u = v$ for a given $v \in \mathbb{B}$, and was treated in [LS] (where earlier references are given). In modern terminology, if a solution exists then v is called a **coboundary**. A necessary condition is

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obviously $\sup_N \|\sum_{j=0}^N T^j v\| < \infty$, but this condition is not sufficient in general (see [LS] for cases when it is sufficient, e.g., for T a dual operator in a dual Banach space).

Let K be a compact Hausdorff space. A **Markov operator** on $C(K)$ is a positive contraction T with $T1 = 1$. Since $C(K)^*$ is the space of finite Baire measures, $P(x, A) = T^*\delta_x(A)$ defines a transition probability [Fo],[K, p.177], and T can be extended to all bounded Baire functions (by $Tf(x) = \int f(y)P(x, dy)$). A non-empty closed subset A is called **absorbing** if T^*m is supported in A whenever $m \in C(K)^*$ is supported in A (equivalently, as $T1 = 1$, if $T1_A \geq 1_A$). T is called **irreducible** if the only absorbing set is K.

MAIN THEOREM: *Let K be a compact Hausdorff space, and let T be an irreducible Markov operator on $C(K)$. If $g \in C(K)$ satisfies $\sup_N \|\sum_{j=0}^N T^j g\| < \infty$, then (and only then) there exists $f \in C(K)$ with $(I - T)f = g$.*

Proof: We first prove the theorem for K metric.

For any bounded measurable function ϕ, let $\tilde{\phi} = \inf\{h \in C(K): \phi \leq h\}$. Then $\tilde{\phi}$ is clearly upper semi-continuous and bounded. Since K is metric, there is a decreasing sequence $\{h_n\}$ of continuous functions with $\tilde{\phi} = \inf h_n$ [W, p. 50]. Also we have that $\tilde{\phi}$ is the smallest upper semi-continuous function above ϕ. Since we have $T\tilde{\phi} = \lim Th_n \geq T\phi$, the upper semi-continuity of $T\tilde{\phi}$ implies $T\tilde{\phi} \geq \widetilde{T\phi}$.

Let $\phi = \limsup_N \sum_{j=0}^N T^j g$, which is bounded by our assumption. Then

$$T\phi \geq \limsup_N \sum_{j=0}^N T^{j+1} g = \limsup_N (\sum_{j=0}^{N+1} T^j g - g) = \phi - g.$$

Hence $T\tilde{\phi} \geq \widetilde{T\phi} \geq \widetilde{(\phi - g)} = \tilde{\phi} - g$. Applying the same argument to the function $-g$, we obtain $\tilde{\psi}$ bounded upper semi-continuous with $T\tilde{\psi} \geq \tilde{\psi} + g$. By adding we have $T(\tilde{\phi} + \tilde{\psi}) \geq \tilde{\phi} + \tilde{\psi}$. Let $h = \alpha - (\tilde{\phi} + \tilde{\psi})$, where $\alpha = \sup(\tilde{\phi} + \tilde{\psi})$. Since $T1 = 1$, we have $Th \leq h$. Clearly h is lower semi-continuous, with $\inf\{h(x): x \in K\} = 0$. By compactness, the closed set $B = \{x: h(x) = 0\}$ is not empty. Let $A = B^c = \{x: h(x) > 0\}$, and $A_n = \{x: h(x) \geq \frac{1}{n}\}$. Since $h \geq \frac{1}{n} 1_{A_n}$, we have $h \geq Th \geq \frac{1}{n} T1_{A_n}$ so for $x \in B$ we obtain $T1_{A_n}(x) = 0$. Hence for $x \in B$ we have $T1_A(x) = \lim_n T1_{A_n}(x) = 0$, which means $T1_A \leq 1_A$ everywhere. Hence the closed non-empty set B is absorbing [K, p.118],[Fo]. Since T is irreducible, $B = K$ [K, p. 179], so h is identically 0. This yields $\tilde{\phi} = \alpha - \tilde{\psi}$.

Since the right hand side is lower semi-continuous and the left hand side is upper semi-continuous, $\tilde{\phi}$ is continuous, and satisfies $T\tilde{\phi} \geq \tilde{\phi} - g$. We put $f = \tilde{\phi}$.

Let μ be an invariant probability for T (such exist [K, p. 178]). Our assumption yields $\int g d\mu = 0$. Since $Tf - f + g \geq 0$ and $\int (Tf - f + g) d\mu = 0$, the open set $\{x\colon (Tf - f + g)(x) > 0\}$ has μ-measure 0. Irreducibility implies that the support of μ is K (since the support of an invariant probability is a closed absorbing set), so $Tf - f + g = 0$ everywhere. The proof for K metric is now complete.

The proof for the general case is obtained by reduction to the metric case as in [J]: Let C' be the closed T-invariant subalgebra of $C(K)$ generated by $\{T^j g\colon j \geq 0\}$ and the constants. Then C' is isomorphic to $C(K')$, where K' is the set of equivalence classes of points of K modulo the subalgebra, which is compact Hausdorff for the quotient topology (see also [DuSch, p. 274]). K' is metrizable, since $C(K')$ is separable by the construction of C' in [J]. The T-invariance of C' yields a Markov operator T' on $C(K')$, which is also irreducible. We now apply the metric case to T', and use the isomorphism to obtain that there is $f \in C'$ with $(I - T)f = g$.

COROLLARY 1.1 ([GH, p. 135]): *Let θ be a minimal continuous map of a compact Hausdorff space K into itself (i.e., every orbit is dense). Then $g \in C(K)$ is of the form $f - f \circ \theta$ if (and only if) $\sup_N \|\sum_{j=0}^N g \circ \theta^j\| < \infty$.*

Proof: Minimality of θ is equivalent to irreducibility of T defined by $Tf(x) = f(\theta x)$ [K].

COROLLARY 1.2: *Let T be a uniquely ergodic Markov operator on $C(K)$ with the invariant probability supported on all of K, and let $g \in C(K)$. Then there exists $f \in C(K)$ with $(I - T)f = g$ if (and only if) $\sup_N \|\sum_{j=0}^N T^j g\| < \infty$.*

Proof: T is irreducible, since any closed absorbing set supports an invariant probability.

Remarks: 1. A uniquely ergodic T is mean ergodic [K, p. 178], so if $g = h - Th$, the function $f = h - \int h d\mu$, which satisfies $(I - T)f = g$, is obtained by taking the (uniform) limit of

$$\frac{1}{N}\sum_{n=1}^{N}\sum_{j=0}^{n-1} T^j g = h - \frac{1}{N}\sum_{n=1}^{N} T^n h.$$

If T is irreducible and not uniquely ergodic, convergence need not hold (take h for which the averages do not converge in norm). However, the proof of the theorem would be the same if we defined $\phi = \limsup_N \frac{1}{N}\sum_{n=1}^{N}\sum_{j=0}^{n-1} T^j g$. It is this definition which will be generalized in the sequel in order to extend the theorem to more general semi-groups.

2. The second part of the proof of the theorem shows that if the equation $(I-T)f = g$ has a solution, the solutions are in the closed T-invariant subalgebra generated by $\{T^j g\colon j \geq 0\}$ and the constants. If T is not uniquely ergodic, there may be no solution in the closed linear manifold generated by $\{T^j g\colon j \geq 0\}$ (since it should then be given as in remark 1).

3. Example 3 in [LS] shows that unique ergodicity alone (without irreducibility) is not sufficient for the theorem to be true.

4. Corollary 1.2 was first proved in [Sc], using a similar approach, but relying heavily on the unique ergodicity.

5. Previous proofs of corollary 1.1 ([GH], [LS]) do not generalize to prove our theorem.

6. After seeing the proof of the Main Theorem, J.P. Conze informed us that he had used a similar approach (unpublished) in order to prove Corollary 1.1 (in the metric case) without using Zorn's lemma.

We now give an application of Corollary 1.2. Let G be a locally compact σ-compact group. For f bounded measurable and m a probability on G, we define the **convolution** operator by $m * f(x) = \int f(xt)dm(t)$. In particular, the right translate of a function f is given by $\delta_t * f(x) = f(xt)$. Note that $m * f$ is continuous for f continuous. A bounded continuous function f on G is called **(weakly) almost-periodic** if the set of right translates $\{\delta_t * f\colon t \in G\}$ is (weakly) conditionally compact in $C(G)$. The space of all almost-periodic functions is denoted by $AP(G)$. For the sake of completeness we include the proof of the following Lemma from [DL].

LEMMA 1.3: *Let G be a locally compact σ-compact group, and let m be an adapted probability on G (i.e., the closed subgroup generated by its support is all of G). If f is weakly almost periodic, then $\frac{1}{n}\sum_{k=1}^{n} m^k * f$ converges uniformly to a constant.*

Proof: Since $\{\delta_t * f\colon t \in G\}$ is conditionally compact in the weak topology of $C(G)$, its closed convex hull $\overline{\text{co}}\{\delta_t * f\colon t \in G\}$ is weakly compact, and contains all the convolutions $\mu * f$ for any probability μ. Hence the sequence of averages $\frac{1}{n}\sum_{k=1}^{n} m^k * f$ is weakly sequentially compact, so by the Yosida–Kakutani mean ergodic theorem [K, p. 72] converges in the norm of $C(G)$ (uniformly). The limit function g is weakly almost-periodic, and satisfies $m * g = g$. By a result of Ryll-Nardzewski [RN] we have $\delta_t * g = g$ for m-almost every t. But the set $G' = \{t: \delta_t * g = g\}$ is a subgroup of G, and since g is continuous G' is closed. Since $m(G') = 1$, it contains the support S, and as m is adapted, $G' = G$. Hence g is invariant under all translations, so is a constant.

PROPOSITION 1.4: *Let G be a locally compact σ-compact group, and let m be an adapted probability on G. Then $g \in AP(G)$ satisfies $\sup_N \|\sum_{j=0}^N m^j * g\| < \infty$ if and only if there exists $f \in AP(G)$ with $f - m * f = g$.*

Proof: It is known that $AP(G)$ is a Banach algebra, isometrically and order isomorphic to $C(\hat{G})$ for some compact group $\hat{G}$ [Lo, p. 168] (the almost-periodic compactification of G). Let T on $C(\hat{G})$ be the representation of the convolution operator on $AP(G)$. Then T is uniquely ergodic by the previous Lemma. The constant $M(h) = \lim \frac{1}{n} \sum_{k=1}^n m^k * h$ yields the invariant mean on $AP(G)$, which corresponds in $\hat{G}$ to the Haar measure. Hence the unique T-invariant probability is supported on all of $\hat{G}$, and we can apply Corollary 1.2 to T. Using the representation, we obtain exactly the claim of the proposition.

Remarks: 1. The case of G compact (where $AP(G) = C(G)$) is an immediate consequence of Corollary 1.2, and appears in [Sc] (with an application).

2. The space $W(G)$ of weakly almost-periodic functions is also isomorphic to $C(K)$ for some compact Hausdorff space [B, p. 4], and if T on $C(K)$ is the representation of the operator of convolution by m, then it is uniquely ergodic by the Lemma. However, the unique T-invariant probability is not supported on K (the unique invariant mean on $W(G)$ can vanish on non-negative functions in $W(G)$), so we cannot prove the proposition for $W(G)$ instead of $AP(G)$.

Let $B(X, \mathcal{A})$ be the space of bounded measurable functions on a measurable space $(X, \mathcal{A})$, and $P(x, A)$ a transition probability. Though $B(X, \mathcal{A})$ is isometrically and order isomorphic to $C(K)$ for an appropriate compact Hausdorff space K [DuSch, p.274], the Main Theorem cannot be applied, since in general the transferred Markov operator is not irreducible. The following result, communicated to us by R. Wittmann, uses the particular nature of the space for solving Poisson's equation, without using the isomorphism to $C(K)$.

THEOREM 1.5: *Let $P(x, A)$ be a transition probability on a measurable space $(X, \mathcal{A})$, with induced Markov operator P. If $g \in B(X, \mathcal{A})$ satisfies*

$\sup_N \|\sum_{j=0}^N P^j g\| < \infty$, *then (and only then) there exists* $f \in B(X,\mathcal{A})$ *with* $(I-P)f = g$.

Proof: Let $\phi = \limsup_N \sum_{j=0}^N P^j g$, which is bounded by our assumption. Then

$$P\phi \geq \limsup_N \sum_{j=0}^{N} P^{j+1} g = \limsup_N (\sum_{j=0}^{N+1} P^j g - g) = \phi - g.$$

Hence $g + (P-I)\phi \geq 0$, so the sequence $h_n = \sum_{i=0}^n P^i(g + P\phi - \phi)$ is increasing, and

$$\|h_n\| \leq \|\sum_{i=0}^n P^i g\| + \|\sum_{i=0}^n P^i(P-I)\phi\| \leq \|\sum_{i=0}^n P^i g\| + 2\|\phi\|.$$

Hence $h = \lim_n h_n$ is in $B(X,\mathcal{A})$, and

$$Ph = \lim_n Ph_n = \lim_n \sum_{i=1}^{n+1} P^i(g + P\phi - \phi) = h - (g + P\phi - \phi).$$

Hence the bounded function $f = h + \phi$ satisfies $(I-P)f = g$.

Remarks: 1. The property $P1 = 1$ can be replaced in the proof by $\sup_n \|P^n\| < \infty$.

2. The same method can be used to prove the existence of a solution in the case of a one-parameter semi-group of Markov operators (the continuous time case).

3. The above theorem (and the previous remark) answer questions from [LS] and [KL].

2. Extensions of the results to semi-groups

In this section we extend the Main Theorem to irreducible Markov operator (anti)-representations of more general semi-groups. The Gottschalk–Hedlund theorem [GH] will be completely generalized, but for general Markov operators some structure on the semi-group will be needed.

A non-empty closed subset is **absorbing** for a semi-group of Markov operators on $C(K)$ if it is absorbing for each operator in the semi-group. The semi-group is **irreducible** if its only closed absorbing set is K. A semi-group of continuous self-maps of K is **minimal** if every orbit is dense; minimality is equivalent to irreducibility of the induced semi-group of Markov operators.

Let S be a semi-group, and Σ a σ-algebra of subsets of $\mathcal{S}$. When Σ is the collection of all subsets of $\mathcal{S}$, we'll write $\mathcal{S}$ instead of $(\mathcal{S}, \Sigma)$. An **action** of $(\mathcal{S}, \Sigma)$ in the compact Hausdorff space K is a family $\{\theta_s: s \in \mathcal{S}\}$ of continuous self-maps of K satisfying $\theta_{s_1 s_2}x = \theta_{s_1}(\theta_{s_2}x)$ for $x \in K$, such that $(s,x) \longrightarrow \theta_s x$ is measurable on $\mathcal{S} \times K$. The Markov operators $T_s f(x) = f(\theta_s x)$ induced on $C(K)$ by the action yield an **anti-representation** of $\mathcal{S}$ (i.e., $T_{s_1 s_2} = T_{s_2}T_{s_1}$), which is **bimeasurable**, i.e., for any $f \in C(K)$ the map $(s,x) \to T_s f(x)$ is measurable on $\mathcal{S} \times K$. Hence, $\{T_s\}$ is weakly measurable: for any $f \in C(K)$ and $\mu \in C(K)^*$ the function $s \to \int T_s f \, d\mu$ is a Σ-measurable function [N, III.2]. (If we had defined the action by operating on the *right*, we'd have an anti-action in our notation, and a representation in $C(K)$.) Any anti-representation $\{T_s\}$ by Markov operators in $C(K)$ yields a representation $\{T_s^*\}$ in $C(K)^*$, the space of Baire measures on K.

Definition: Let $\{T_s\}$ be an anti-representation of $\mathcal{S}$ by Markov operators in $C(K)$. A function $g(s,x)$ is called a (continuous) **coboundary** for $\{T_s\}$ if there exists $f \in C(K)$ with $g(s,x) = f(x) - T_s f(x)$. It is necessarily continuous in x (and bounded). If $\{T_s\}$ is bimeasurable, a coboundary is measurable on $\mathcal{S} \times K$.

For $f \in C(K)$, define $g_s = f - T_s f$. Then

$$g_{s_1 s_2} = f - T_{s_2}(T_{s_1} f) = f - T_{s_2} f + T_{s_2}(f - T_{s_1} f) = g_{s_2} + T_{s_2} g_{s_1}.$$

Definition: A (bounded) **cocycle** for $\{T_s\}$ as above is a bimeasurable (bounded) real function $g(s,x)$, continuous in x for fixed $s \in \mathcal{S}$, which satisfies $g_{s_1 s_2} = g_{s_2} + T_{s_2} g_{s_1}$, where $g_s(x) = g(s,x)$. When $\{T_s\}$ is induced by an action $\{\theta_s\}$, a cocycle is characterized by $g(s_1 s_2, x) = g(s_2, x) + g(s_1, \theta_{s_2} x)$.

For a single Markov operator T, $g(n,x)$ is a cocycle if and only if it is of the form $g(n,x) = \sum_{j=0}^{n-1} T^j g_1(x)$.

Cocycles (usually of group actions) appear (sometimes in disguised form) in various problems of measurable [Sm] and topological [FKeMSe] dynamics. Group-valued (generalized) cocycles (in measurable dynamics) are treated in [Z].

The method of proof of the Gottschalk–Hedlund theorem [GH] can be extended to any semi-group to yield the following result. For completeness we give the arguments.

THEOREM 2.1: *Let $\{\theta_s\}$ be an action of a semi-group $\mathcal{S}$ in the compact Hausdorff space K which is minimal. If $g(s,x)$ is a bounded cocycle, then there exists*

$f \in C(K)$ such that $g(s,x) = f(x) - f(\theta_s x)$ for $s \in \mathcal{S}$, $x \in K$ (i.e, g is a coboundary).

Proof: Let $Y = K \times \mathbb{R}^1$ with the product-topology. For $s \in \mathcal{S}$ define σ_s on Y by

$$\sigma_s y = (\theta_s x,\, \alpha + g(s,x)) \quad \text{for } y = (x,\alpha), \quad x \in K, \quad \alpha \in \mathbb{R}^1.$$

We show that $\{\sigma_s\}$ is an action in Y: If $s_1, s_2 \in \mathcal{S}$, then

$$\sigma_{s_1}(\sigma_{s_2} y) = \sigma_{s_1}(\theta_{s_2} x, \alpha + g(s_2,x))$$

$$= (\theta_{s_1}\theta_{s_2} x, \alpha + g(s_2,x) + g(s_1, \theta_{s_2} x) = (\theta_{s_1 s_2} x, \alpha + g(s_1 s_2, x)\ ) = \sigma_{s_1 s_2} y.$$

Take any $y_0 \in Y$. Since $g(s,x)$ is bounded, the orbit $O(y_0) = \{\sigma_s y_0 \colon s \in \mathcal{S}\}$ is precompact in Y; therefore, its closure $\overline{O(y_0)}$ is a compact $\{\sigma_s\}$-invariant subset of Y. A standard argument with Zorn's lemma yields that $\overline{O(y_0)}$ contains a closed non-empty set M which is minimal for $\{\sigma_s\}$ (i.e., a closed $\{\sigma_s\}$-invariant set with no proper subsets satisfying these properties).

We claim that M must actually be the graph of some bounded real function on K, i.e., $M = \{(x, f(x)) \colon x \in K\}$ for some bounded $f(x)$. Indeed, the projection of M into K, being $\{\theta_s\}$-invariant, must be all of K. Fix $x_0 \in K$. If $(x_0, \alpha_i) \in M$, $i = 1,2$, with $\beta = \alpha_1 - \alpha_2 \neq 0$, then $M + \beta := \{(x, \alpha+\beta) \colon (x,\alpha) \in M\}$ is also $\{\sigma_s\}$-invariant. But this implies the invariance of the intersection $M \cap (M + \beta)$, which contradicts the minimality of M. Hence, for $x_0 \in K$, there is exactly one point of the form (x_0, α) in M.

Since f is bounded and its graph is closed, it must be continuous. For $s \in \mathcal{S}$, $x \in K$, we obtain, using the invariance of M:

$$(\theta_s x, f(\theta_s x)) = \sigma_s(x, f(x)) = (\theta_s x, f(x) + g(s,x)).$$

Therefore, $f(\theta_s x) = f(x) + g(s,x)$. This concludes the proof.

Unlike the previous result, additional assumptions are needed for the case of semi-group representations by Markov operators.

We assume that for any $s \in \mathcal{S}$ and $A \in \Sigma$ the sets As and $As^{-1} := \{t \in \mathcal{S} \colon ts \in A\}$ are in Σ (these assumptions, clearly satisfied for $\Sigma =$ all subsets, are used for general mean ergodic theorems [K, p. 221]; the second one means that multiplication on the right is a measurable map). We call a σ-finite measure m on Σ **right translation-invariant** if $m(As) = m(A)$ for any set $A \in \Sigma$ and any $s \in \mathcal{S}$. A sequence $\{F_n\}$ of measurable subsets of $\mathcal{S}$ with $m(F_n) < \infty$, such that $m(F_n \Delta F_n s)/m(F_n) \to 0$ for every $s \in \mathcal{S}$, is called a **Følner sequence**.

THEOREM 2.2: *Let $(\mathcal{S}, \Sigma)$ be a semi-group as above, which has has a right translation invariant σ-finite measure and a Følner sequence. Let K be a compact metric space, and let $\{T_s\}$ be a bimeasurable anti-representation of $\mathcal{S}$ by Markov operators in $C(K)$, which is irreducible. If $g(s,x)$ is a bounded cocycle, then there exists $f \in C(K)$ such that $g(s,x) = f(x) - T_s f(x)$ for $s \in \mathcal{S}$, $x \in K$ (i.e. g is a coboundary).*

Proof: The definition of ϕ in the proof of our Main Theorem uses the order in $\mathbb{N}$. However, the alternative in remark 1 after the theorem suggests that we define

$$\phi(x) = \limsup_n \frac{1}{m(F_n)} \int_{F_n} g(t,x)dm(t).$$

For fixed $x \in K$, the integral (well defined by the measurability of $g(s,x)$) is a continuous function in x by Lebesgue's dominated theorem, since g is bounded (and K is metric). Hence ϕ is bounded measurable on K, with $\|\phi\|_\infty \leq \|g\|_\infty$.

For $s \in \mathcal{S}$ we have a transition probability $P_s(.,.)$ which yields T_s, so by Fatou's lemma and Fubini's theorem we obtain

$$\begin{aligned} T_s\phi(x) &= \int_K [\limsup_n \frac{1}{m(F_n)} \int_{F_n} g(t,y)dm(t)]P_s(x,dy) \\ &\geq \limsup_n \int_K [\frac{1}{m(F_n)} \int_{F_n} g(t,y)dm(t)]P_s(x,dy) \\ &= \limsup_n \frac{1}{m(F_n)} \int_{F_n} T_s g_t(x)dm(t) \\ &= \limsup_n \frac{1}{m(F_n)} \int_{F_n} [g_{ts}(x) - g_s(x)]dm(t) \\ &= \limsup_n \frac{1}{m(F_n)} \int_{F_n} g(ts,x)dm(t) - \ g_s(x). \end{aligned}$$

By the right translation invariance, we obtain for each $x \in K$ (see [K, p.222])

$$\int_{F_n} g(ts,x)dm(t) = \int_{F_n s} g(t,x)dm(t).$$

Using the definition of Følner sequences, we obtain

$$\frac{1}{m(F_n)} \left| \int_{F_n} g(t,x)dm(t) - \int_{F_n s} g(t,x)dm(t) \right| \leq \|g\|_\infty m(F_n \Delta F_n s)/m(F_n) \to 0.$$

We therefore obtain $T_s\phi \geq \phi - g_s$ for any $s \in \mathcal{S}$. The same arguments as in the proof of our Main Theorem show that $\tilde{\phi}$ is continuous, and $T_s\tilde{\phi} \geq \tilde{\phi} - g_s$ for each s.

We apply the same argument to $-g(s,x)$, which is also a bounded cocycle, and obtain $\tilde{\psi} \in C(K)$ with $T_s\tilde{\psi} \geq \tilde{\psi} + g_s$ for each s. By adding we have $T_s(\tilde{\phi} + \tilde{\psi}) \geq \tilde{\phi} + \tilde{\psi}$ for each s. Irreducibility yields (as in the proof of the Main Theorem) that $\tilde{\phi} + \tilde{\psi} = c$, so $T_s\tilde{\phi} \leq \tilde{\phi} - g_s$ for each s. This shows that $(I - T_s)\tilde{\phi} = g_s$ for every $s \in \mathcal{S}$.

Remark: In the proof of the Main Theorem, the last step was simplified by the existence of an invariant probability, which in our general case would require also left amenability.

COROLLARY 2.3: *Let $\mathcal{S}$ be a second countable topological semi-group with Σ the σ-algebra of its Borel sets, and assume $(\mathcal{S}, \Sigma)$ satisfies the assumptions of the theorem. Let K be a compact Hausdorff space, and let $\{T_s\}$ be a strongly continuous anti-representation of $(\mathcal{S}, \Sigma)$ by Markov operators in $C(K)$, which is irreducible. Then every bounded separately continuous cocycle $g(s,x)$ is a coboundary.*

Proof: The function $(s,x) \to T_s f(x)$ is jointly continuous, since

$$\begin{aligned} |T_s f(x) - T_{s_0} f(x_0)| &\leq |T_s f(x) - T_{s_0} f(x)| + |T_{s_0} f(x) - T_{s_0} f(x_0)| \\ &\leq \|T_s f - T_{s_0} f\| + |T_{s_0} f(x) - T_{s_0} f(x_0)|. \end{aligned}$$

Hence we have bimeasurability, and the previous theorem applies if K is metrizable.

When K is not metrizable, we look, as in the proof of the Main Theorem, at C', the closed $\{T_s\}$-invariant subalgebra of $C(K)$ generated by $\{g_s\colon s \in \mathcal{S}\}$ and the constants. We show that C' is separable. Let $\{s_n\}$ be a dense countable subgroup of $\mathcal{S}$. Let $E_1 = \{1\} \cup \{T_{s_j} g_{s_n}\}$. We proceed as in [J] — inductively; let $E_{k+1} = E_k \cup \{T_{s_j} h\colon h$ is a finite product of functions of $E_k\}$. (We added E_k since we do not assume a unit in $\mathcal{S}$.) Then $\{E_k\}$ is an increasing sequence of countable sets, so its union E_∞ is countable. By definition, products of two functions in E_k are in E_{k+1}, so E_∞ is closed under products, hence it generates a separable subalgebra of $C(K)$, say $\tilde{C}$. Since each E_k is closed under the action of every T_{s_n}, the subalgebra $\tilde{C}$ is also invariant for each T_{s_n}. By second countability, for $s \in \mathcal{S}$ there is $s_{n_j} \to s$, so by strong continuity $\tilde{C}$ is also T_s-invariant. For $s \in \mathcal{S}$, take again $s_{n_j} \to s$. By the separate continuity of g, we have $g_{s_{n_j}} \to g_s$ pointwise, and the boundedness of g and Lebesgue's theorem yield $g_{s_{n_j}} \to g_s$ weakly in $C(K)$. Since closed convex sets are weakly closed, $g_s \in \tilde{C}$, proving $C' = \tilde{C}$. The theorem now follows from the metric case, as in our Main Theorem.

Example: Let K be a compact Hausdorff space, and let $\{T_t\colon t \geq 0\}$ be an irreducible weakly continuous semi-group of Markov operators on $C(K)$, with infinitesimal generator D. If $g \in C(K)$ satisfies $\sup_{r\geq 0} \| \int_0^r T_t g\, dt\| < \infty$, then there exists $f \in C(K)$ with $Df = g$.

Proof: For the definition of a strongly continuous one-parameter semi-group and its infinitesimal generator we refer the reader to [DuSch], [HiP]. Since $\{T_t\colon t \geq 0\}$ is assumed weakly continuous at 0, it is strongly continuous at $t \geq 0$ [HiP, p. 324].

The function $h(r, x) = \int_0^r T_t g(x)\, dt$ is a cocycle, and the previous corollary applies to yield a function $f \in C(K)$ such that $T_s f - f = -\int_0^s T_t g\, dt$ for every $s > 0$. Hence

$$s^{-1}(T_s f - f) = -s^{-1} \int_0^s T_t g\, dt \to_{s\to 0+} -g$$

by the continuity at 0 of the semi-group. This shows $f \in \text{Domain}(D)$ and $Df = -g$.

3. Coboundaries of semi-group actions in measurable spaces

In this section we look at actions in the space of all measurable functions and in L_∞ spaces. Though these spaces are isometrically and order isomorphic to $C(K)$ for an appropriate compact Hausdorff space K [DuSch, p. 274], the previous results cannot be applied, since in general the transferred Markov operator semi-group is not irreducible. However, the particular nature of these spaces will enable us to obtain some results (without using the isomorphism to $C(K)$). We combine the approach of [LS] with the method of the previous section.

Let $B(X, \mathcal{A})$ be the space of all bounded measurable functions on a measurable space $(X, \mathcal{A})$. In the previous definitions of an action, a coboundary and a cocycle, we replace K by X and continuous functions by bounded measurable ones, and omit the (now meaningless) continuity assumptions.

THEOREM 3.1: *Let $(\mathcal{S}, \Sigma)$ be a semi-group which has a right translation invariant σ-finite measure and a Følner sequence, and let $\{\theta_s\}$ be an action of $(\mathcal{S}, \Sigma)$ in the measurable space $(X, \mathcal{A})$. Then every bounded cocycle is a coboundary.*

Proof: We use the previous notations. Let $g(s,x)$ be a bounded cocycle. As in Theorem 2.2, we define

$$\phi(x)=\limsup_n \frac{1}{m(F_n)}\int_{F_n} g(t,x)dm(t).$$

By boundedness and bimeasurability of g, the integral is well-defined, and is $\mathcal{A}$-measurable [N, III.2]. For $s \in \mathcal{S}$ we have

$$\phi(\theta_s x)=\limsup_n \frac{1}{m(F_n)}\int_{F_n} g(t,\theta_s x)dm(t).$$

The computations in the proof of Theorem 2.2 now yield $\phi(\theta_s x)=\phi(x)-g(s,x)$ for any $x \in X,\quad s \in \mathcal{S}$.

Remarks: 1. Theorem 3.1 does not need any irreducibility assumptions. However, unlike Theorem 2.1, some additional assumptions on the semi-group are still needed.

2. The theorem applies when $\mathcal{S}$ is a locally compact σ-compact amenable group with its Baire σ-algebra.

3. We do not know if the theorem can be extended to general Markov operators.

If $(X,\mathcal{A},\mu)$ is a finite measure space, and the action in Theorem 3.1 is by non-singular transformations, the proof applies also in $L_\infty(X,\mathcal{A},\mu)$. However, since the action induces a weak-* measurable anti-representation in $L_\infty(\mu)$, the result can be obtained even for Markov operators on L_∞, as a consequence of the following general abstract result.

THEOREM 3.2: *Let $(\mathcal{S},\Sigma)$ be a semi-group which has a right translation invariant σ-finite measure and a Følner sequence, and let $\{R_s\}_{s\in\mathcal{S}}$ be a weakly measurable representation of $(\mathcal{S},\Sigma)$ by contractions in a Banach space $\mathbb{B}$. Let $g(s)$ be a bounded weak-* measurable function from $\mathcal{S}$ to $\mathbb{B}^*$ (i.e., $\langle g(s),v\rangle$ is measurable for any $v \in \mathbb{B}$), satisfying*

$$g(s_1s_2)=g(s_2)+R^*_{s_2}g(s_1)\quad \forall s_1,s_2 \in \mathcal{S}.$$

Then there exists $f \in \mathbb{B}^*$ such that $g(s)=f-R^*_s f$ for every $s \in \mathcal{S}$.

Proof: As before, m is a right translation invariant measure and $\{F_n\}$ a Følner sequence. Fix $v \in \mathbb{B}$. Then $f_n(v)=\frac{1}{m(F_n)}\int_{F_n}\langle g(t),v\rangle dm(t)$ is well-defined by weak-* measurability and boundedness of g, and f_n is clearly an element of $\mathbb{B}^*$, with $\|f_n\| \leq \|g\|_\infty$. For $s \in \mathcal{S}$ we have

$$\begin{aligned}\langle R_s^* f_n, v\rangle &= \frac{1}{m(F_n)} \int_{F_n} \langle g(t), R_s v\rangle dm(t) = \frac{1}{m(F_n)} \int_{F_n} \langle R_s^* g(t), v\rangle dm(t) \\ &= \frac{1}{m(F_n)} \int_{F_n} \langle g(ts) - g(s), v\rangle dm(t) \\ &= \frac{1}{m(F_n)} \int_{F_n} \langle g(ts), v\rangle dm(t) - \langle g(s), v\rangle.\end{aligned}$$

Let f be a weak-* limit point of $\{f_n\}$. Computations as in the proof of Theorem 2.2 yield $R_s^* f = f - g(s)$ for any s.

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